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SOME LIMIT THEOREMS ON SET-FUNCTIONS

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1. Introduction. In a recent paper [1] we have considered two limit theorems on set-functions in an abstract set. The first theorem is a generalization of the theorem on differentiation on a net, the net being replaced by an increasing sequence of σ-fields. The second theorem is a sort of counterpart of the first, the sequence of σ-fields being now decreasing. The theorems had presented themselves as generalizations of known theorems on integration of functions of infinitely many variables.

When publishing our paper we were not aware that essentially equivalent results had already been published by Doob [2], though in a form in which the close connection with the known results on functions of infinitely many variables is less apparent. There is, however, the difference, that while Doob considers point-functions, which amounts to assume the set-functions continuous with respect to the given measures, we have, in the first theorem, made this assumption only for the contractions to the σ-fields of the sequence, whereas the set-function itself was allowed to contain a singular part.

The object of the present paper is to prove generalizations of the two theorems, in which no assumptions regarding continuity of the set-functions with respect to the measures are made. Thus we obtain two theorems which are completely analogous. For this purpose only slight changes in the former proofs are required, but for the convenience of the reader we give the proofs in detail. Actually the generalization makes the proofs more conspicuous.

2. Derivative of a set-function with respect to a measure. In addition to the definitions and theorems stated at the beginning of [1] we shall use the following fundamental theorem:

Let E be a set containing at least one element, and μ a measure in E with domain $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}$ and $\mu(E) = 1$. Then to any bounded, completely additive set-function φ with domain $\mathfrak{F}$ there exists a μ-integrable function f with $[f] = E$ such that the μ-continuous part φ_c of φ is the indefinite integral of f, i. e.

$$\varphi_c(A) = \int_A f(x)\,\mu(dE)$$

for any $A \varepsilon \mathfrak{F}$, and that the positive and negative parts of the μ-singular part φ_s of φ for any $A \varepsilon \mathfrak{F}$ are determined by

$$\varphi_s^+(A) = \varphi(A\,[f = +\infty]) \quad \text{and} \quad \varphi_s^-(A) = \varphi(A\,[f = -\infty]).$$

Any such function f will be called a *derivative of φ with respect to μ.*

It is easily seen that if f_0 is a derivative of φ with respect to μ then a μ-measurable function f is a derivative of φ with respect to μ if and only if $\mu([f \neq f_0]) = 0$ and $\varphi(A) = 0$ for any sub-set $A \varepsilon \mathfrak{F}$ of $[f \neq f_0]$.

A μ-measurable function f is a derivative of φ with respect to μ if and only if it satisfies the following conditions: For an arbitrary (finite) number a we have $\varphi(A) \leq a\mu(A)$ for any sub-set $A \varepsilon \mathfrak{F}$ of $[f \leq a]$ and $\varphi(A) \geq a\mu(A)$ for any sub-set $A \varepsilon \mathfrak{F}$ of $[f \geq a]$.

The necessity of the first condition is plain, for since $A\,[f = +\infty] = 0$ we have

$$\varphi(A) = \int_A f(x)\,\mu(dE) + \varphi(A\,[f = -\infty]) \leq \int_A a\mu(dE) + 0 = a\mu(A).$$

The necessity of the second condition is proved analogously.

The sufficiency of the conditions is well known from the proof of the above mentioned theorem.

3. The two limit theorems. Let E be a set containing at least one element, and μ a measure in E with domain $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}$ and $\mu(E) = 1$. Let φ be a bounded, completely additive set-function with domain $\mathfrak{F}$.

Let $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$ be a sequence of σ-fields contained in $\mathfrak{F}$, such that $E \varepsilon \mathfrak{F}_n$ for all n. Let μ_n and φ_n denote the contractions of μ and φ to $\mathfrak{F}_n$, and let f_n denote a derivative of φ_n with respect to μ_n.

The first limit theorem now states:

If $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \subseteq \cdots$ *then the functions*

$$\underline{f} = \liminf_n f_n \quad \textit{and} \quad \overline{f} = \limsup_n f_n$$

are derivatives of φ' *with respect to* μ', *where* μ' *and* φ' *are the contractions of* μ *and* φ *to the smallest* σ*-field* $\mathfrak{F}'$ *containing all* $\mathfrak{F}_n$.

In the particular case in which φ_n for every n is μ_n-continuous, this theorem is equivalent to the first limit theorem of our previous paper [1].

The second limit theorem states:

If $\mathfrak{F}_1 \supseteq \mathfrak{F}_2 \supseteq \cdots$ *then the functions*

$$\underline{f} = \liminf_n f_n \quad \textit{and} \quad \overline{f} = \limsup_n f_n$$

are derivatives of φ' *with respect to* μ', *where* μ' *and* φ' *are the contractions of* μ *and* φ *to the largest* σ*-field* $\mathfrak{F}'$ *contained in all* $\mathfrak{F}_n$.

In the particular case in which φ is μ-continuous, and hence φ_n for every n is μ_n-continuous, this theorem is equivalent to the second limit theorem of [1].

4. Proof of the first limit theorem. Since $\underline{f}$ and $\overline{f}$ evidently are μ'-measurable it will according to § 2 be sufficient to prove the inequalities

$$\varphi(HA) \leq a\mu(HA) \quad \text{and} \quad \varphi(KA) \geq a\mu(KA)$$

for any $A \varepsilon \mathfrak{F}'$, when $H = [\underline{f} \leq a]$ and $K = [\overline{f} \geq a]$ for an arbitrary number a.[1]

In order to prove the first inequality we put

[1] For if A is a sub-set of $[\underline{f} \leq a]$ or $[\overline{f} \leq a]$ we have $HA = A$, and if A is a sub-set of $[\underline{f} \geq a]$ or $[\overline{f} \geq a]$ we have $KA = A$.

$$H_n = [\inf_p f_{n+p} < a_n]$$

and

$$H_{np} = \begin{cases} [f_{n+1} < a_n] \text{ for } p = 1 \\ [f_{n+1} \geqq a_n, \cdots, f_{n+p-1} \geqq a_n, f_{n+p} < a_n] \text{ for } p > 1, \end{cases}$$

where $a_1, a_2, \cdots$ denotes a (strictly) decreasing sequence of numbers converging towards a. Then $H_{np} \varepsilon \mathfrak{F}_{n+p}$ and $H_{np} \subseteqq [f_{n+p} < a_n]$. Clearly (for a given n) no two of the sets H_{np} have elements in common, and $H_n = \sum_p H_{np}$. Further $H_1 \supseteqq H_2 \supseteqq \cdots$ and $H = \mathfrak{D}_n H_n$. Now, if A belongs to the field $\mathfrak{G} = \mathfrak{S}_n \mathfrak{F}_n$, we shall have $A \varepsilon \mathfrak{F}_n$ for all $n \geqq$ (some) n_0; hence $H_{np} A \varepsilon \mathfrak{F}_{n+p}$ for $n \geqq n_0$ and all p. We therefore have

$$\begin{aligned} \varphi(H_n A) &= \sum_p \varphi(H_{np} A) = \sum_p \varphi_{n+p}(H_{np} A) \\ &\leqq \sum_p a_n \mu_{n+p}(H_{np} A) = \sum_p a_n \mu(H_{np} A) = a_n \mu(H_n A). \end{aligned}$$

Since $H_1 A \supseteqq H_2 A \supseteqq \cdots$ and $HA = \mathfrak{D}_n H_n A$, we have $\mu(HA) = \lim_n \mu(H_n A)$ and $\varphi(HA) = \lim_n \varphi(H_n A)$. We therefore obtain $\varphi(HA) \leqq a\,\mu(HA)$.

We now define a set-function $\varkappa$ on $\mathfrak{F}'$ by placing

$$\varkappa(A) = a\,\mu(HA) - \varphi(HA).$$

Clearly $\varkappa$ is bounded and completely additive. Moreover, since $\varphi(HA) \leqq a\,\mu(HA)$ for $A \varepsilon \mathfrak{G}$, the contraction of $\varkappa$ to $\mathfrak{G}$ is non-negative. Since $\mathfrak{F}'$ is the smallest σ-field containing $\mathfrak{G}$ this implies that the set-function $\varkappa$ itself is non-negative, i. e. the inequality $\varphi(HA) \leqq a\,\mu(HA)$ is valid for all $A \varepsilon \mathfrak{F}'$.

The inequality $\varphi(KA) \geqq a\,\mu(KA)$ is proved analogously.

5. Corollaries of the first limit theorem. If in particular $\mathfrak{F}' = \mathfrak{F}$, we have $\mu' = \mu$ and $\varphi' = \varphi$, so that the first limit theorem contains statements about the set-function φ itself.

Even if $\mathfrak{F}' \subset \mathfrak{F}$, we may, however, by means of the following general remark concerning derivatives, under a certain additional assumption, deduce results regarding the set-function φ.

Let E, μ, $\mathfrak{F}$, and φ be as in § 2, and let μ' and φ' denote the contractions of μ and φ to a σ-field $\mathfrak{F}' \subset \mathfrak{F}$, such that $E \varepsilon \mathfrak{F}'$.

Let f' denote a derivative of φ' with respect to μ'. Suppose, that *to any set $A \varepsilon \mathfrak{F}$ there exist sets $B \varepsilon \mathfrak{F}'$ and $C \varepsilon \mathfrak{F}'$ such that $B \subseteq A \subseteq C$ and $\mu(C - B) = 0$. Then*

(i) *if φ is non-negative f' is also a derivative of φ with respect to μ;*

(ii) *in any case the indefinite integral of f' with respect to μ is the μ-continuous part of φ.*

Proof. (i) Let $A \varepsilon \mathfrak{F}$, and let B and C be corresponding sets according to the assumption. On placing $H = [f' \leqq a]$ and $K = [f' \geqq a]$ we have

$$\varphi(HA) \leqq \varphi(HC) \leqq a\mu(HC) = a\mu(HA)$$

and

$$\varphi(KA) \geqq \varphi(KB) \geqq a\mu(KB) = a\mu(KA).$$

(ii) The statement follows easily by application of (i) to the set-functions φ^+ and $-\varphi^-$.

Our assumption does not imply that f' for an arbitrary φ is a derivative of φ with respect to μ. This is shown by the following example:[1]

Let $\mathfrak{F}$ consist of all sub-sets of a set E of three elements a, b, and c, and let $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(\{c\}) = 0$, and $\varphi(\{a\}) = 0$, $\varphi(\{b\}) = 1$, $\varphi(\{c\}) = -1$. Let $\mathfrak{F}'$ consist of all sets containing either both or none of the elements b and c. Then the function $f' = 0$ is a derivative of φ' with respect to μ', but not of φ with respect to μ.

6. Proof of the second limit theorem. In this case $\mathfrak{F}' = \underset{n}{\mathfrak{D}}\mathfrak{F}_n$ Since $\underline{f}$ and $\overline{f}$ are μ_n-measurable for all n, they are μ'-measurable; according to § 2 it is therefore sufficient to prove the inequalities

$$\varphi(HA) \leqq a\mu(HA) \quad \text{and} \quad \varphi(KA) \geqq a\mu(KA)$$

for any $A \varepsilon \mathfrak{F}'$, when $H = [\inf\limits_n f_n < a]$ and $K = [\sup\limits_n f_n > a]$ for an arbitrary number a.[2]

[1] A corresponding example in our previous paper ([1], pp. 12—13) is wrong as it stands. The above example shows how it may be rectified.

[2] For if A is a sub-set of $[\underline{f} \leqq a]$ or $[\overline{f} \leqq a]$, it is a sub-set of $[\inf\limits_n f_n < a + \varepsilon]$ for any $\varepsilon > 0$, hence $\varphi(A) \leqq (a + \varepsilon)\,\mu(A)$ and consequently $\varphi(A) \leqq a\mu(A)$. Similarly, if A is a sub-set of $[\underline{f} \geqq a]$ or $[\overline{f} \geqq a]$, it is a sub-set of $[\sup\limits_n f_n > a - \varepsilon]$ for any $\varepsilon > 0$, hence $\varphi(A) \geqq (a - \varepsilon)\,\mu(A)$ and consequently $\varphi(A) \geqq a\mu(A)$.

In order to prove the first inequality it is sufficient to prove that if for an arbitrary n we put

$$H_n = [\min_{p \leq n} f_p < a]$$

we have $\varphi(H_n A) \leq a\mu(H_n A)$ for any $A \varepsilon \mathfrak{F}'$. For $H_1 \subseteq H_2 \subseteq \cdots$ and $H = \mathfrak{S}_n H_n$. Hence $\mu(HA) = \lim_n \mu(H_n A)$ and $\varphi(HA) = \lim_n \varphi(H_n A)$.

To prove the inequality $\varphi(H_n A) \leq a\mu(H_n A)$ we put

$$H_{np} = \begin{cases} [f_p < a,\ f_{p+1} \geq a, \cdots, f_n \geq a] \text{ for } p < n \\ [f_n < a] \text{ for } p = n. \end{cases}$$

Then $H_{np} \varepsilon \mathfrak{F}_p$ and $H_{np} \subseteq [f_p < a]$. Moreover $H_n = \sum_{p \leq n} H_{np}$. Since $A \varepsilon \mathfrak{F}_p$ for any p this implies

$$\varphi(H_n A) = \sum_{p \leq n} \varphi(H_{np} A) = \sum_{p \leq n} \varphi_p(H_{np} A)$$

$$< \sum_{p \leq n} a\mu_p(H_{np} A) = \sum_{p \leq n} a\mu(H_{np} A) = a\mu(H_n A).$$

The inequality $\varphi(KA) \geq a\mu(KA)$ is proved analogously.

References.

[1] E. Sparre Andersen and B. Jessen. Some limit theorems on integrals in an abstract set. D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. 22, no. 14 (1946).

[2] J. L. Doob. Regularity properties of certain families of chance variables. Trans. Amer. Math. Soc. 47 (1940), pp. 455—486.

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